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Flows in a rotating spherical shell: the equatorial case

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It is well known that the widely used powerful geostrophic equations that single out the vertical component of the Earth's rotation cease to be valid near the equator. Through a vorticity and an angular momentum analysis on the sphere, we show that if the flow varies on a horizontal scale L smaller than $(Ha)^{1/2}$ (where H is a vertical scale of motion and a the Earth's radius), then equatorial dynamics must include the effect of the horizontal component of the Earth's rotation. In equatorial regions, where the horizontal plane aligns with the Earth's rotation axis, latitudinal variations of planetary angular momentum over such scales become small and approach the magnitude of its radial variations proscribing, therefore, vertical displacements to be freed from rotational constraints. When the zonal flow is strong compared to the meridional one, we show that the zonal component of the vorticity equation becomes $(2\boldsymbol{\Omega}\cdot\nabla)u_1 = (g/\rho_0)(\partial\rho/a\partial\theta)$. This equation, where θ is latitude, expresses a balance between the buoyancy torque and the twisting of the full Earth's vorticity by the zonal flow u_1. This generalization of the mid-latitude thermal wind relation to the equatorial case shows that u_1 may be obtained up to a constant by integrating the 'observed' density field *along the Earth's rotation axis* and not along gravity as in common mid-latitude practice. The simplicity of this result valid in the finite-amplitude regime is not shared however by the other velocity components.

Vorticity and momentum equations appropriate to low frequency and predominantly zonal flows are given on the equatorial β-plane. These equatorial results and the mid-latitude geostrophic approximation are shown to stem from an exact generalized relation that relates the variation of dynamic pressure along absolute vortex lines to the buoyancy field. The usual hydrostatic equation follows when the aspect ratio $\delta = H/L$ is such that $\tan\theta/\delta$ is much larger than one. Within a boundary-layer region of width $(Ha)^{1/2}$ and centred at the equator, the analysis shows that the usually neglected Coriolis terms associated with the horizontal component of the Earth's rotation must be kept.

Finally, some solutions of zonally homogeneous steady equatorial inertial jets are presented in which the Earth's vorticity is easily turned upside down by the shear flow and the correct angular momentum '$\Omega r^2\cos^2(\theta) + u_1 r\cos(\theta)$' contour lines close in the vertical–meridional plane.

1. Introduction

The balance between Coriolis forces and pressure gradients discovered at the beginning of the 19th century has now received enough experimental verification to become one of the foundations of present day Meteorology and Oceanography. More precisely the 'so called' geostrophic equilibrium happens when the component of the

Coriolis force in the horizontal plane is nearly balanced by the pressure gradient. If $\boldsymbol{k}$ denotes a unit vector parallel to gravity, it reads

$$f\boldsymbol{k}\times\boldsymbol{u} = -\frac{\nabla p}{\rho}, \tag{1}$$

where f, the Coriolis parameter, is $2\Omega\sin\theta$ (θ being the latitude). In this relationship, the projection of the Earth's rotation vector $\boldsymbol{\Omega}$ onto the local vertical is assumed to capture the dominant effects of rotation. When (1) is used with the hydrostatic relation:

$$-\frac{\partial P}{\partial r} - g\rho = 0, \tag{2}$$

where r is the radial coordinate, the thermal wind equation can be derived:

$$f\frac{\partial \boldsymbol{u}}{\partial r} = \boldsymbol{g}\times\frac{\nabla\rho}{\rho}, \tag{3}$$

an expression which relates the shear of the horizontal velocity field in the radial direction to the horizontal gradient of density. Oceanographers, who lack direct pressure measurements, make considerable use of (3) to deduce the velocity field from the observed temperature and salinity fields up to an arbitrary function of horizontal position. This arbitrary function of position, the so-called reference level of ocean currents, may be determined using additional data and/or dynamics. Sverdrup (1947) noted that the equation for the divergence of the geostrophic flow derived from (1) could provide this missing constant of integration:

$$-v\frac{\cot\theta}{r} + \frac{\partial w}{\partial r} = 0. \tag{4}$$

By integrating (4) from a deep level (where w is assumed to be zero) to the bottom of the upper Ekman layer where the vertical velocity is equal to the Ekman pumping, a relation, known since as the Sverdrup relation, is obtained which links the *meridional transport of water to the wind stress curl.* Furthermore, through an integration with respect to longitude, the zonal transport can be estimated up to an arbitrary constant that was fixed by Sverdrup to ensure no normal flow at the *eastern* side of an ocean basin.

Presented as such, equations (1)–(4) seem relatively *ad hoc*. They may be given a firmer foundation with the help of scale analysis. Assuming that the motion occurs with a well-defined horizontal scale L and vertical scale H, equations (1)–(3) appear to be consistent approximations of the hydrodynamic equations for situations of slow motions or, more precisely, small Rossby number U/fL, and small aspect ratio H/L. Indeed when the rotation is rapid and the flow contained in thin shells, vertical accelerations and vertical velocities are much smaller than their horizontal counterparts and horizontal accelerations are also much smaller than the Coriolis accelerations. The chain of arguments leading to this result appears in Phillips' (1963) review. Phillips introduces, in fact, two classes of geostrophic motion: the first kind occurs when the horizontal scale is much smaller than the Earth's radius, in which case the horizontal divergence of velocity is of the order of the Rossby number, an approximation that leads directly to the quasi-geostrophic set of equations, the pressure being identified with a stream function. The second relates to planetary motions for which the divergence of the geostrophic velocity becomes large: this is the situation to which

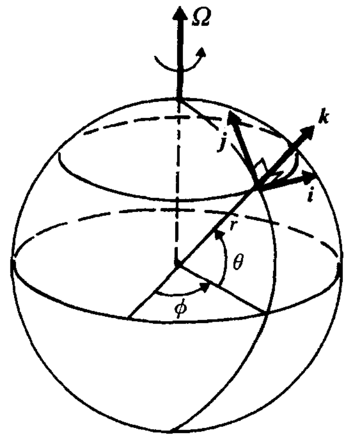


FIGURE 1. Illustration of the geometry of the local spherical coordinate system (ϕ, θ, r) respectively (longitude, latitude, distance from the Earth's centre). (i, j, k) are unit vectors in the east, north and local vertical directions which are associated to (1, 2, 3) in the text.

equation (4) applies. The point examined in the present study is that all the relations (1)–(4) are appropriate approximations for mid-latitude flows and cease to be valid in the vicinity of the equator. Indeed the thermal wind relation (3) produces singular flows at the equator unless the density gradient vanishes there. If it vanishes, the relation loses any *predictive* power for inference of the velocity field. Similarly, relation (4) shows that the divergence of the geostrophic flow increases without limit at the equator and this is clearly not compatible with the usual small vertical velocities allowed in thin-shell geometries. As noted by Phillips 'a consistent approximation procedure has not yet been developed for low latitudes'. That this equatorial singularity occurs is no surprise since, with the approximations made, the fluid is effectively not rotating at the equator. Geostrophy, if it does apply in this region, must relate to the true direction of the Earth's rotation axis and not to its vertical projection, and Taylor columns, if present, must be aligned in that direction. It is the objective of this work to examine first under what conditions this generalized geostrophic equilibrium can occur for a fluid contained in a thin rapidly rotating spherical shell and second to propose realistic modifications of that equilibrium, valid for slow large-scale zonal flows in the ocean–atmosphere context. To do this, it is useful to look at the problem in terms of vorticity.

Consider for instance, the thermal wind equation (3). As pointed out by Lighthill (1966) this relation is a vorticity equation that expresses a balance between the torque of the buoyancy forces and the twisting of the vertical component of the Earth's vorticity by the vertical shear of the horizontal flow. This gives the clue that the search for low-latitude approximations may proceed in a fruitful way from a vorticity viewpoint. As it is the density that is observed in geophysical flows, this will have the advantage of freeing the approximation procedure from *a priori* assumptions about the pressure forces. The reader may think that the twisting of the meridional component of the Earth's vorticity vector by horizontal shear flows may be the missing term at low latitudes to balance the buoyancy torques. However, as the subsequent analysis shows, care must be exercised because this term is small and many other terms may enter the approximated vorticity equations at the same level of smallness.

The small parameter of the problem that will guide our ordering of terms is the *global* Rossby number $\epsilon = U/2\Omega L$ (with U and L referring to horizontal velocity and horizontal lengthscale respectively), a choice that avoids any reference to a specific latitude range. Because the Earth spins rapidly ($2\Omega = 1.45\ 10^{-4}$ rad s^{-1}), ϵ values of 10% requires lengthscales in excess of 35 km and 700 km respectively for the ocean and atmosphere given respective flows speeds of 0.5 m s^{-1} and 10 m s^{-1}. These horizontal scales are sufficiently small that an expansion of the variables in powers of ϵ is thought to capture the essential dynamics of the energy-containing flows at larger scales in both fluids. A second fundamental ordering parameter is the aspect ratio $\delta = H/L$ of vertical over horizontal scales whose upper bound is respectively of order 0.1 and 0.01 for the oceanic and atmospheric cases based on the depths of the ocean and troposphere and the above minimal horizontal scales for a 10% value of ϵ. No *a priori* ordering of these two parameters is imposed in this study because ϵ and δ, although both small, are not widely different. Let us examine, however, for its own sake, the linear approximations, i.e. the case ϵ very much smaller than δ.

When the geopotential surfaces are assumed spherical and the gravity vector radial and of constant value (see Gill 1982) the steady linearized momentum equations are

$$-2\Omega \sin\theta\, u_2 + 2\Omega\cos\theta\, u_3 = -\frac{1}{\rho_0 r\cos\theta}\frac{\partial P}{\partial\phi}, \tag{5a}$$

$$2\Omega\sin\theta\, u_1 = -\frac{1}{\rho_0 r}\frac{\partial P}{\partial\theta}, \tag{5b}$$

$$-2\Omega\cos\theta\, u_1 = -\frac{\partial P}{\rho_0\partial r} - \frac{g\rho}{\rho_0}. \tag{5c}$$

In (5) ρ_0 is a constant reference density and ρ its variable part, so that the Boussinesq approximation is made. The coordinate system is shown in figure 1. Latitude θ is measured from the equatorial plane and longitude ϕ from an appropriate meridian increasing eastwards. The unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are respectively oriented eastwards, northwards and upwards and indices 1, 2, 3 refer to components oriented in these respective directions. In this reference frame, the rotation vector $\boldsymbol{\Omega}$ has components (0, $\Omega\cos\theta$, $\Omega\sin\theta$). If the pressure is eliminated by taking the curl of (5), the vorticity equation is obtained:

$$(2\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\, u_1 = \frac{g}{\rho_0 r}\frac{\partial\rho}{\partial\theta}, \tag{6a}$$

$$(2\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\, u_2 - \frac{2\Omega\cos\theta}{r}u_3 = -\frac{g}{\rho_0 r\cos\theta}\frac{\partial\rho}{\partial\phi} \tag{6b}$$

$$(2\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\, u_3 + \frac{2\Omega\cos\theta}{r}u_2 = 0, \tag{6c}$$

where the operator $2\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}$ is $2\Omega((\cos\theta/r)(\partial/\partial\theta)\ldots + \sin\theta(\partial/\partial r)\ldots)$. In the horizontal plane, the twisting of the full Earth's rotation vector by the fluid velocities balances the buoyancy torques on the right in (6*a*) and (6*b*) while the vertical vorticity equation (6*c*) is the appropriate generalization of (4) when the true Earth's rotation axis is considered. One of the results of the study that follows is to show, in the geophysical context of the ocean–atmosphere system, that (5*b*), (5*c*) and (6*a*) remain valid at finite amplitude in equatorial regions provided that the zonal component of the flow dominates the meridional one. Because this last assumption is realistic, (6*a*) generalizes the thermal wind: *zonal equatorial flows can be determined from a knowledge of the meridional density gradients.* The other equations are strongly modified for finite values of ϵ and we show that the full zonal momentum equation must be considered in the low-frequency limit at low latitudes, a complexity that projects onto the meridional and vertical components of the vorticity equation. This makes it impossible to infer the meridional and vertical velocity from the density field in a simple manner. The discussion of this result revolves around the validity of the hydrostatic approximation and we show that the Coriolis acceleration terms associated with the horizontal component of the Earth's rotation must be kept in an equatorial boundary layer of width $(Ha)^{1/2}$.

The neglect of these terms, known in the literature as the 'traditional approximation' is a subject of long history. Phillips (1966) proposed a set of equations appropriate for large-scale rotating flows and showed that a form of the angular momentum

$$\Gamma = \Omega r^2 \cos^2(\theta) + u_1 r \cos(\theta) \tag{7}$$

satisfying the conservation principle

$$\frac{D\Gamma}{Dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial \phi} \tag{8}$$

was maintained without these Coriolis terms, provided the varying radius r in (7) was replaced by the constant mean value a of the Earth's radius, i.e. the so-called shallow-water approximation. For set (1)–(2), Γ approximates to $\Omega a^2 \cos^2(\theta)$ without violating the conservation principle. For set (5), the radial variation of the planetary angular momentum has to be kept to retrieve the neglected Coriolis terms, and the correct Γ then is $\Omega r^2 \cos^2(\theta)$. Veronis (1968) cautioned about indiscriminate use of the Phillips' set of equations in equatorial regions. Indeed without making this traditional approximation, Stern (1963) had already found axisymmetric equatorial inertial modes of low frequency (of order $2\Omega(h/r)^{1/2}$ where h is the fluid thickness). These modes that resulted from a finite number of reflections of inertial waves between the inner and outer spherical shells (Bretherton 1964), were later confirmed by Israelis's (1972) numerical calculations. The subject was taken up again by Miles (1974) was examined the singularities of the Laplace tidal equation at critical latitudes and introduced a novel formulation for the tidal problem that kept the usually neglected Coriolis terms and provided a uniformly valid approximation as δ went to zero. The often-given argument that the traditional approximation is justified when the Brunt–Väisälä frequency is much larger than 2Ω comes from a consideration of the dispersion relation for inertial–internal gravity plane waves and is obviously incorrect for motions of scale large enough to feel the spherical shape of the bounding surface. The complete problem of finding the modes of oscillation within spherical shells is notoriously difficult owing to the non-separability of the solutions and still open.

In §2 the complete vorticity equations appropriate for a sphere in rotation are derived and scaled for the case of rapid rotation and thin shells. In §3, it is shown how a modified thermal wind equation may be used in low latitudes to determine zonal velocities from the meridional density gradients. The simplicity of this result does not apply in other directions. We give consistent sets of momentum and vorticity equations that govern the behaviour of low-frequency large-scale predominantly zonal flows on the equatorial β-plane to leading orders in ϵ, δ. Both mid-latitude and low-latitude cases may be cast into a general framework suitable for improved determination of the low-frequency part of the circulation from density observations and simple criteria are proposed to qualify possible uses of the hydrostatic 'traditional' approximation in §4. Finally a class of solutions of low-frequency inertial equatorial jets is discussed in the §5. When ϵ becomes of order δ, the horizontal relative vorticity associated with a zonal jet becomes comparable with the Earth's vorticity and near the equator the direction of the absolute vorticity vector can easily depart from the direction of the Earth's rotation axis. Indeed given realistic geophysical values for such equatorial jets, we show that the absolute vorticity and angular momentum contours form closed loops in the meridional vertical plane. The possibilities opened by this unique situation are briefly explored. We do not discuss specific applications in the present paper but we feel that some of our results are relevant to a variety of situations on rapidly rotating planets in the limit of small but finite aspect ratio and Rossby number. As the thickness of the fluid relative to the radius of the planet becomes larger, the equatorial dynamics emphasized here will appear over an increasing range of latitudes.

2. The vorticity equation

Although at the heart of many interpretations of the dynamics of atmospheric and oceanic flows, it appears that no discussion of the spherical vorticity equations in component form has appeared so far. Instead, considerable attention has been given to Ertel's potential vorticity, a particular combination of vorticity and density, which is conserved following a fluid parcel in the absence of irreversible processes. The fluid is supposed to be incompressible, gravity is radial, and the Boussinesq approximation is made. This latter choice is more appropriate for the ocean than for the atmosphere but it is not necessary at this stage of the analysis to be exhaustive and appropriate modifications are straightforward. Under the above conditions the momentum equations are

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+2\boldsymbol{\Omega}\times\boldsymbol{u}=-\frac{\boldsymbol{\nabla}p}{\rho_0}+\frac{\rho}{\rho_0}\boldsymbol{g}, \tag{9a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=0. \tag{9b}$$

If the curl of (9a) is taken, the vorticity equation in vectorial form becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\xi}+2\boldsymbol{\Omega})=(\boldsymbol{\xi}+2\boldsymbol{\Omega})\cdot\boldsymbol{\nabla}\boldsymbol{u}+\frac{\boldsymbol{\nabla}\rho}{\rho_0}\times\boldsymbol{g}, \tag{10}$$

where $\boldsymbol{\xi}=\boldsymbol{\nabla}\times\boldsymbol{u}$. This states that the absolute vorticity $(\boldsymbol{\xi}+2\boldsymbol{\Omega})$ of a fluid parcel is modified both by the stretching and twisting actions of the velocity field and by the action of the buoyancy torques. Equation (3) is a particular subset of (10): it is what remains of the horizontal components of (10) when the relative vorticity vector is neglected and the operator $2\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}$ is approximated by $2\Omega\sin\theta\,\partial/\partial r$. Although quite justified at mid-latitudes, these approximations become dubious when θ becomes small. What is less obvious is that (4) is also what remains of the vertical component of (10) under the same two conditions. To improve upon these approximations and obtain expressions valid for the whole sphere, a scaling procedure is required and applied to each component of (10) in what follows. The forms of (10) in the zonal, meridional and vertical direction are respectively

$$\frac{\mathrm{D}\xi_1}{\mathrm{D}t}+\frac{\tan\theta}{r}(\xi_1u_2-u_1\xi_2)+\frac{1}{r}(u_1\xi_3-\xi_1u_3)=(\boldsymbol{\xi}+2\boldsymbol{\Omega})\cdot\boldsymbol{\nabla}u_1-\frac{g}{\rho_0 r}\frac{\partial\rho}{\partial\theta}, \tag{11a}$$

$$\frac{\mathrm{D}\xi_2}{\mathrm{D}t}+\frac{1}{r}(u_2\xi_3-\xi_2u_3)-\frac{2\Omega\cos\theta u_3}{r}=(\boldsymbol{\xi}+2\boldsymbol{\Omega})\cdot\boldsymbol{\nabla}u_2+\frac{g}{\rho_0 r\cos\theta}\frac{\partial\rho}{\partial\phi}, \tag{11b}$$

$$\frac{\mathrm{D}\xi_3}{\mathrm{D}t}+\frac{2\Omega\cos\theta}{r}u_2=(\boldsymbol{\xi}+2\boldsymbol{\Omega})\cdot\boldsymbol{\nabla}u_3. \tag{11c}$$

The gradient operator that appears in (11) is

$$\nabla = \left(\frac{1}{r\cos\theta}\frac{\partial}{\partial\phi}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial r}\right)$$

and the vorticity components are related to velocity components as

$$\xi_1 = \frac{1}{r}\frac{\partial u_3}{\partial\theta} - \frac{1}{r}\frac{\partial}{\partial r}(ru_2),$$

$$\xi_2 = \frac{1}{r}\frac{\partial}{\partial r}(ru_1) - \frac{1}{r\cos\theta}\frac{\partial u_3}{\partial\phi},$$

$$\xi_3 = \frac{1}{r\cos\theta}\left[\frac{\partial u_2}{\partial\phi} - \frac{\partial}{\partial\theta}(u_1\cos\theta)\right],$$

while the continuity equation is

$$\frac{1}{r\cos\theta}\left(\frac{\partial u_1}{\partial\phi} + \frac{\partial}{\partial\theta}(u_2\cos\theta)\right) + \frac{1}{r^2}\frac{\partial(u_3 r^2)}{\partial r} = 0.$$

Because the unit vectors are a function of position on the sphere, numerous metric terms appear in these equations. As will be shown, most of them are unimportant under the shallow-water approximation, a notable exception being the second term in the vertical vorticity equation (11*c*). This is the familiar β-term ($\beta = 2\Omega\cos\theta/r$) that plays such a crucial role in the dynamics of slow motions on the Earth.

To find out if the simplified equations (6) have any value and how they need to be modified under realistic conditions, the flow variables in the full equations (11) are scaled. Because the focus of the present work is on equatorial regions, the anisotropy of equatorial flows in introduced at the outset. To achieve this, two independent horizontal scales L_θ, L_ϕ in the meridional and zonal directions respectively are chosen and their ratio $\alpha(= L_\theta/L_\phi)$ is an additional parameter of the problem. Assuming that each term in the continuity equation and material derivative is possibly important, the scales of the velocities (u_1, u_2, u_3) are chosen as $(U, \alpha U, \alpha\delta U)$ where δ, the aspect ratio, is H/L_θ, H being the relevant vertical lengthscale, which is usually different from the depth of the fluid. Similarly the scales of the vorticities (ξ_1, ξ_2, ξ_3) are chosen as $(U\alpha/H, U/H, U/L_\theta)$. Finally the density scale $\rho_0 2\Omega L_\theta U/gH$ is that appropriate for large-scale flows dominated by the Earth's rotation. The new non-dimensional variables (primed) then become

$$u_1' = \frac{u_1}{U}, \quad u_2' = \frac{u_2}{\alpha U}, \quad u_3' = \frac{u_3}{\alpha\delta U},$$

$$\xi_1' = \frac{\xi_1}{U\alpha/H}, \quad \xi_2' = \frac{\xi_2}{U/H}, \quad \xi_3' = \frac{\xi_3}{U/L_\theta},$$

$$\rho' = \frac{gH}{\rho_0 2\Omega U L_\theta}\rho,$$

$$\frac{\partial}{\partial t'} = \frac{L_\phi}{U}\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial\theta'} = L_\theta\frac{\partial}{r\partial\theta},$$

$$\frac{\partial}{\partial\phi'} = L_\phi\frac{\partial}{r\partial\phi}, \quad \frac{\partial}{\partial r'} = H\frac{\partial}{\partial r}.$$

Introducing these changes of variables into (11) produces the following non-dimensional equations (after dropping the primes):

$$\alpha^2\epsilon\frac{D\xi_1}{Dt}+\epsilon\gamma\tan\theta(\alpha^2\xi_1 u_2-u_1\xi_2)+\epsilon\delta\gamma(u_1\xi_3-\alpha^2\xi_1 u_3) = (\epsilon\boldsymbol{\xi}+\boldsymbol{\Omega}^*)\cdot\boldsymbol{\nabla}u_1-\frac{\partial\rho}{\partial\theta}, \quad (12a)$$

$$\epsilon\frac{D\xi_2}{Dt}+\epsilon\delta(\gamma u_2\xi_3-\xi_2 u_3)-\delta^2\gamma\cos\theta u_3 = (\epsilon\boldsymbol{\xi}+\boldsymbol{\Omega}^*)\cdot\boldsymbol{\nabla}u_2+\frac{1}{\cos\theta}\frac{\partial\rho}{\partial\phi}, \quad (12b)$$

$$\epsilon\frac{D\xi_3}{Dt}+\gamma\cos\theta u_2 = (\epsilon\boldsymbol{\xi}+\boldsymbol{\Omega}^*)\cdot\boldsymbol{\nabla}u_3. \quad (12c)$$

In these expressions the variations of r have been neglected (to order H/r) and as a consequence r has been replaced by its constant value a. Furthermore, the variations of the angle θ should in principle be scaled as $(L_\theta/a)\theta'$ but for economy this has not been introduced at this stage in the trigonometrical terms although a local expansion of (12) at a given latitude would require this. The scaled Earth's vorticity vector Ω^* is

$$\boldsymbol{\Omega}^* = \begin{pmatrix} 0 \\ \delta\cos\theta \\ \sin\theta \end{pmatrix}$$

so that the stretching–twisting operator that appears in (12) is

$$(\epsilon\boldsymbol{\xi}+\boldsymbol{\Omega}^*)\cdot\boldsymbol{\nabla} = \alpha^2\epsilon\frac{\xi_1}{\cos\theta}\frac{\partial}{\partial\phi}+(\epsilon\xi_2+\delta\cos\theta)\frac{\partial}{\partial\theta}+(\epsilon\xi_3+\sin\theta)\frac{\partial}{\partial r}. \quad (12d)$$

Expression (12) shows that the various approximations of the full vorticity equation depend upon four independent parameters: the global Rossby number $\epsilon = U/2\Omega L_\theta$; the aspect ratio $\delta = H/L_\theta$; the planetary ratio $\gamma = L_\theta/a$; the anisotropy ratio $\alpha = L_\theta/L_\phi$.

In scaling the angular momentum (7) as $\Gamma' = \Gamma/\Omega a^2$ with the above non-dimensional variables, one obtains

EG $$\Gamma' = \underset{\text{(I)}}{\cos^2(\theta)}+\underset{\text{(II)}}{2\delta\gamma\cos^2(\theta)r'}+\underset{\text{(III)}}{2\epsilon\gamma(1+\delta\gamma r')\cos(\theta)u_1}, \quad (13)$$

where r has been scaled like $a(1+\delta\gamma r')$. A term $\delta^2\gamma^2\cos^2(\theta)r'^2$ of order $(H/a)^2$ has been neglected for consistency with the approximations in (12). Term (I) represents the main planetary term depending only on latitude and which leads to the traditional Coriolis term in the zonal momentum equation. Term (II) is the radially dependent angular momentum part leading to the vertical-velocity-related Coriolis term in the zonal momentum equation, while terms (III) are the relative contributions.

Before turning to new expressions valid at low latitudes, some well-known mid-latitude results are recovered from (12), a derivation that is perhaps complementary to that obtained from a momentum perspective (see for instance Pedlosky 1987).

(*a*) *Geostrophic motion of the first kind*

This is the situation described by the β-plane approximation at mid-latitudes. The horizontal flow is isotropic and the motions remain of a scale small compared to the Earth's radius, so that $\delta \ll 1$, $\epsilon \ll 1$, $\gamma \ll 1$, $\alpha \sim O(1)$. The trigonometrical terms can be expanded around a central latitude θ_0 in powers of γ after expressing the dimensional angle θ as

$$\theta = \theta_0+\gamma\theta'.$$

To leading orders in δ, ϵ, γ, (12*a*) and (12*b*) reduce to the familiar thermal wind relations

$$\sin\theta_0 \frac{\partial u_1}{\partial r} = \frac{\partial \rho}{\partial \theta}, \tag{14a}$$

$$\sin\theta_0 \frac{\partial u_2}{\partial r} = -\frac{1}{\cos\theta_0}\frac{\partial \rho}{\partial \phi}. \tag{14b}$$

On the other hand, (12*c*) implies $\partial u_3/\partial r = 0$. In a formal expansion procedure in powers of ϵ this means that the vertical velocity u_3 is smaller than thought *a priori* and is in fact of order $U\epsilon\alpha\delta$. When rescaling the vertical velocity that way, the vertical vorticity equation (12*c*) becomes to leading order

$$\epsilon\frac{\mathrm{D}\xi_3}{\mathrm{D}t} + \gamma\cos\theta_0\, u_2 = \epsilon\sin\theta_0\frac{\partial u_3}{\partial r}, \tag{14c}$$

an expression in which $\mathrm{D}/\mathrm{D}t$ reduces now to the horizontal advection operator. Because the divergence of the horizontal flow is of order ϵ, a stream function can be introduced to express u_1, u_2 and ξ_3 in terms of that stream function. This is consistent with (14*a*, *b*) because the sine and cosine coefficients in these equations are now constant and evaluated at a central latitude θ_0, from which the motions do not depart significantly ($\gamma \ll 1$). The equations (14) are then equivalent to the classical quasi-geostrophic equations on the Cartesian β-plane tangent to the sphere at θ_0. The consistent angular momentum Γ then takes the form

$$\Gamma = \underset{\text{(I)}}{\cos^2(\theta_0) - 2\cos(\theta_0)\sin(\theta_0)\gamma\theta'} \underset{\text{(II)}}{-2\cos^2(\theta_0)\gamma^2\theta'^2 + \gamma^2\theta'^2} \underset{\text{(III)}}{+ 2\epsilon\gamma\cos(\theta_0)\, u_1},$$

where the cosine coefficients have been expanded up to the second order since Γ represents an integral of the zonal momentum equation. To infer zero- and first-order formulations, the velocities have to be expanded in ϵ, i.e. $u_1 = u_1^{(0)} + \epsilon u_1^{(1)}$ and $u_2 = \mathrm{D}\theta'/\mathrm{D}t = u_2^{(0)} + \epsilon u_2^{(1)}$. The ratio of (III) and (II) compares the relative angular momentum to the planetary one and is $\epsilon/\gamma\cos(\theta_0) = U/(\beta L^2)$, the familiar term that measures the relative importance of advection of relative vorticity against advection of planetary vorticity.

(*b*) *Geostrophic motion of the second kind*

The only difference with the previous limit is that the motion occurs truly on a planetary scale so that γ is now of order 1 and the trigonometrical terms cannot be approximated. When $\delta \ll 1$, $\epsilon \ll 1$, $\gamma \sim O(1)$, and $\alpha \sim O(1)$ the horizontal-vorticity equations (14*a*) and (14*b*) remain valid (with allowed variations for the sine and cosine coefficients) while the vertical-vorticity equation now becomes to leading order

$$\gamma\cos\theta\, u_2 = \sin\theta\frac{\partial u_3}{\partial r}. \tag{15}$$

The resulting set is usually called the planetary geostrophic or thermocline equations in oceanography and have been used to study the thermohaline circulation from both theoretical and observational points of view. This large-scale limit is particularly interesting for the ocean because the internal Rossby radius of deformation is rather small (of order 50 km) and therefore a large range of oceanic scales is permitted, from say 200 to 5000 km, under this limit. One must not forget, however, that the small-scale turbulent motions of the order of the Rossby radius that are filtered out, contain in general most of the relative vorticity and kinetic energy. As has already been mentioned in the introduction, (15) and (4) become singular at low latitudes and a new analysis is needed to study the slow dynamics of equatorial flows. In this mid-latitude planetary limit, the consistent angular momentum reads

$$\Gamma = \cos^2(\theta),$$

an expression which only includes the traditional Coriolis terms in the momentum equations.

3. The low-latitude limit

It has already been mentioned that observed motions at the equator do not favour a particular ordering of the major small parameters of the problem, i.e. global Rossby number ϵ and aspect ratio δ. When θ becomes sufficiently small, so that $\sin\theta$ which is $O(\gamma)$, becomes of order (ϵ, δ), the scaled vorticity equations (12) show that the various terms in the advection of relative vorticity and the stretching–twisting of relative vorticity are all equally important. The problem is rendered intrinsically nonlinear and it appears very difficult to progress. However, observed flows in low latitudes have a very special character: they tend to be oriented rather zonally and be confined in the meridional direction, so that the horizontal shear is dominated by the meridional variation of the zonal flow. With this anisotropy ($\alpha \ll 1$), it will be seen that important simplifications appear that can be used to advantage.

Because the zonal relative vorticity ξ_1 involves the vertical derivative of the meridional velocity which is $O(\alpha)$ smaller than the zonal one, (12*a*) shows that both the advection and the stretching of the zonal relative vorticity ξ_1 are of order α^2 smaller than the other terms. When $\delta \ll 1$, $\alpha \ll 1$, $\gamma \ll 1$ and $\theta \ll 1$, (12*a*) reduces at leading orders to

$$(\epsilon\xi_2 + \delta\cos\theta)\frac{\partial u_1}{\partial\theta} + (\epsilon\xi_3 + \sin\theta)\frac{\partial u_1}{\partial r} = \frac{\partial\rho}{\partial\theta}, \tag{16}$$

an equation, therefore, accurate to $O(\alpha^2, \gamma^2\alpha^2, \delta\gamma, \gamma^2)$, with the ratio of the terms neglected to the smallest terms kept, themselves of $O(\epsilon, \delta)$. Of course, to leading order (16) implies formally that $\partial\rho/\partial\theta$, which is $O(1)$, vanishes. A non-trivial solution needs a rescaling of the density which must be $O(\delta, \epsilon$ or $\gamma)$ smaller than the original mid-latitude value. This rescaling is implicit in the discussion that follows. As soon as $\sin\theta$, which is of $O(\gamma)$, becomes of order ϵ or δ (whichever is larger), equation (16) shows that the zonal buoyancy torque on the right-hand side is balanced not only by the familiar twisting of the vertical component of the Earth's rotation vector, but also by two potentially important additional contributions, the twisting of the horizontal component of the Earth's rotation vector by the horizontal shear flow and the twisting of the component of the relative vorticity vector in the meridional–vertical plane. However, we proceed to show that this second contribution vanishes when α is small. With the notation $\boldsymbol{\xi}^* = (0, \xi_2, \xi_3)$, (16) can be compactly rewritten as

$$(\epsilon\boldsymbol{\xi}^* + \boldsymbol{\Omega}^*)\cdot\nabla u_1 = \frac{\partial\rho}{\partial\theta}. \tag{17a}$$

In much the same way, we obtain for the two other directions at leading order

$$\epsilon\frac{D\xi_2}{Dt} = (\epsilon\boldsymbol{\xi}^* + \boldsymbol{\Omega}^*)\cdot\nabla u_2 + \frac{1}{\cos\theta}\frac{\partial\rho}{\partial\phi}, \tag{17b}$$

$$\epsilon\frac{D\xi_3}{Dt} + \gamma\cos\theta u_2 = (\epsilon\boldsymbol{\xi}^* + \boldsymbol{\Omega}^*)\cdot\nabla u_3, \tag{17c}$$

with an accuracy $O(\alpha^2, \delta\gamma, \delta^2)$. As such, it appears that nothing much has been gained and that equations (17) still retain far too much complexity. However, (16) and (17a) simplify considerably after the important terms in the relative vorticity vector, itself, are considered:

$$\xi_1 = -\frac{\partial u_2}{\partial r} - \delta\gamma\, u_2 + \delta^2 \frac{\partial u_3}{\partial \theta},$$

$$\xi_2 = \frac{\partial u_1}{\partial r} + \delta\gamma\, u_1 - \delta^2\alpha^2 \frac{1}{\cos\theta}\frac{\partial u_3}{\partial \phi},$$

$$\xi_3 = -\frac{\partial u_1}{\partial \theta} + \gamma \tan\theta\, u_1 + \frac{\alpha^2}{\cos\theta}\frac{\partial u_2}{\partial \phi}.$$

This shows readily that to the same accuracy, $O(\delta^2, \alpha^2, \gamma^2, \gamma\delta)$, the non-dimensional relative vorticity vector relates to velocity as

$$\xi_1 = -\frac{\partial u_2}{\partial r}, \quad \xi_2 = \frac{\partial u_1}{\partial r}, \quad \xi_3 = -\frac{\partial u_1}{\partial \theta}. \tag{18}$$

When (18) is introduced into (16), the nonlinear relative vorticity terms cancel out remarkably to produce the zonal vorticity equation:

$$\delta \cos\theta \frac{\partial u_1}{\partial \theta} + \sin\theta \frac{\partial u_1}{\partial r} = \frac{\partial \rho}{\partial \theta}. \tag{19a}$$

To leading orders in (α^2, δ^2, γ^2, $\gamma\delta$, $\gamma\alpha^2$), it appears that the buoyancy torque in the zonal direction is balanced only by the twisting of the full Earth's rotation vector by the zonal velocity, a most simple physical balance which can be rewritten as

$$(\boldsymbol{\Omega}^* \cdot \boldsymbol{\nabla})\, u_1 = \frac{\partial \rho}{\partial \theta}. \tag{19b}$$

This appears to be a uniformly valid approximation over the whole sphere, generalizing the zonal component of the mid-latitude thermal wind equations (3), under the anisotropy assumption. This assumption is the only restriction because the cancellation of the nonlinear terms that has occurred between (16) and (19a) shows that the final expression (19a, b) is valid at any order in ϵ, provided of course that $\alpha^2\epsilon, \epsilon\gamma^2, \epsilon\delta\gamma$ be much less than δ and/or γ as shown by (12a). The global Rossby number ϵ can therefore be larger than δ or γ in this limit of small α. The value of this relation comes from its possible use in the nonlinear regime, a simplicity, however, that is not shared with the other vorticity components. Only the zonal vorticity equation retains the simple linear form (6a) of the generalized geostrophic set of the introduction. The result demonstrates the importance of keeping the full expression for the Coriolis terms in the momentum equations and provides a criterion for the neglect of the horizontal component of the Earth's rotation. At mid-latitudes, θ is order one, α is order one but ϵ is small so that the form of the $\boldsymbol{\Omega}^* \cdot \boldsymbol{\nabla}$ operator shows that the effect of the horizontal component of the Earth's rotation can be neglected if

$$(\tan\theta)/\delta \gg 1,$$

a criterion already quoted by Veronis (1973) from the form of Ertel's potential vorticity. The traditional hydrostatic approximation is well justified for large-scale mid-latitude flow when δ is much smaller than one. Near the equator, however, θ is small ($\theta \approx \gamma\theta'$) and the above criterion becomes

$$(\tan\theta)/\delta \approx \gamma\theta'/\delta \gg 1,$$

where

$$\gamma/\delta = Ha/L_\theta^2. \tag{20}$$

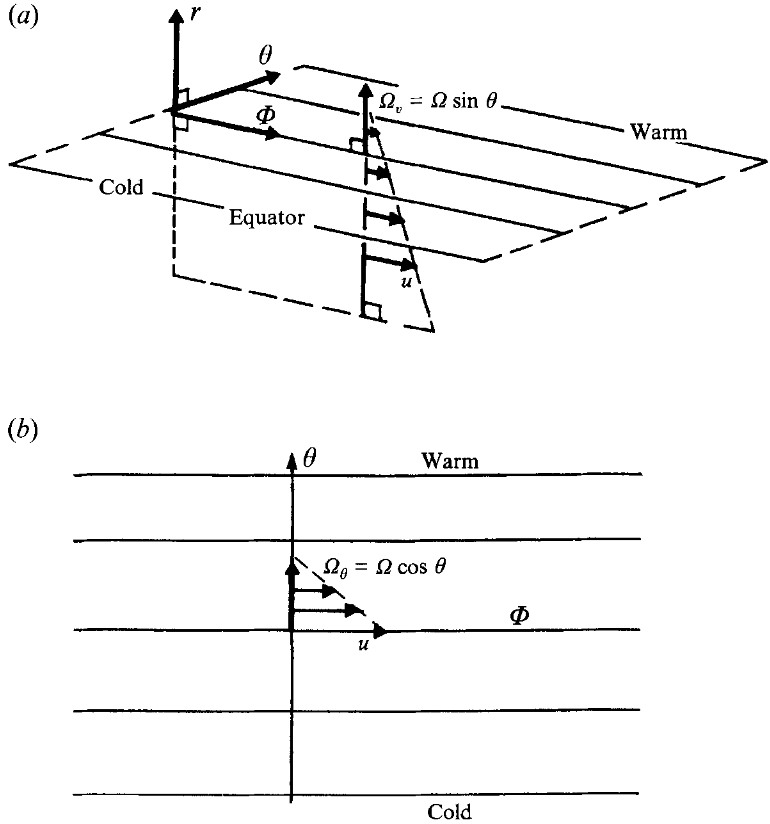


FIGURE 2. Illustration of possible balances in the zonal vorticity equation:

$$2\Omega\cos\theta\frac{\partial u_1}{r\,\partial\theta}+2\Omega\sin\theta\frac{\partial u_1}{\partial r}=\frac{g}{\rho_0}\frac{\partial\rho}{r\,\partial\theta}.$$

(*a*) Baroclinic production of vorticity due to a north–south density gradient compensated by a zonal tilting of the vertical vorticity component $2\Omega_v$. (*b*) Baroclinic production of vorticity due to a north–south density gradient compensated by a zonal tilting of the meridional vorticity component $2\Omega_\theta$. Near the equator both balances have equal strength.

The condition requires the flow to evolve on a scale broader than the natural lengthscale $(Ha)^{1/2}$, which is respectively about 80 km (240 km) given a vertical scale of motion H of order 1 km in the ocean (10 km in the atmosphere). The fact that considerable energy is found at low latitudes in the form of zonal jets on these lateral scales indicates that the traditional approximation should be seriously questioned.

Relation (19*a*, *b*) can be used in a straightforward way. Because it is a first-order quasi-linear equation, the zonal velocity u_1 can be obtained by integrating the observed meridional density gradient along known characteristics which are the straight curves parallel to the Earth's rotation axis. As with the original thermal wind relation, the solution is obtained up to an arbitrary constant, values of u_1 on an initial curve, non-parallel to the rotation axis. Consider for instance, a situation (figure 2) similar to what is found in the ocean with cold water at the equator flanked by warm water of subtropical origin on either side. In the northern hemisphere this produces a buoyancy torque that induces vorticity in the positive eastward direction. Away from the

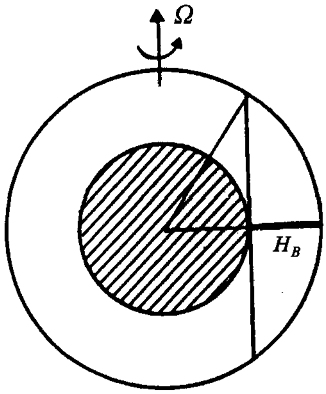


FIGURE 3. Illustration of the equatorial band defined at the base by the cylinder tangent at the equator to the solid spherical Earth of radius a and limited at the top by the spherical shell of radius $a+H_B$. The angle limiting this region for the ocean is around 4° and the arclength subtended by this angle is of order 450 km.

equator, negative vertical shear of the zonal flow acting on the vertical component of $\boldsymbol{\Omega}$ produces a twisting of the right sign to equilibrate the buoyancy torque. Closer to the equator, however, the necessary negative vorticity tendency is presumably carried out increasingly by the twisting of the horizontal component of $\boldsymbol{\Omega}$ by the negative meridional shear of the zonal flow.

These new dynamics should occur in the region visualized by the intersection of the cylinder tangent to the sphere at its bottom (figure 3). The latitudinal extent of this region is given by

$$\cos\theta = a/(a+H_B)$$

or, as H_B/a is small,

$$\theta = (2H_B/a)^{1/2},$$

where H_B is the bottom depth.

In this region, about 2° on each side of the equator for the depth of the ocean and about 4° for the depth of the atmospheric troposphere, the new twisting term of the horizontal component of the Earth's rotation vector becomes of primary importance because the characteristics of (19) are tangent to the spherical surfaces and integration proceeds almost with respect to latitude.

Given u_1 the meridional and vertical components of the vorticity are now known to leading orders from (18). In this case, the meridional and vertical vorticity equations (17*b*) and (17c) become quasi-linear as well for the unknown u_2 and u_3, because the small twisting of the zonal vorticity ξ_1 does not appear at leading order in these equations.

They can be rewritten as

$$\epsilon\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial u_1}{\partial r}\right) = (\epsilon\boldsymbol{\xi}^* + \boldsymbol{\Omega}^*)\cdot\nabla u_2 + \frac{1}{\cos\theta}\frac{\partial\rho}{\partial\phi}, \tag{21a}$$

$$\epsilon\frac{\mathrm{D}}{\mathrm{D}t}\left(-\frac{\partial u_1}{\partial\theta}\right) + \gamma\cos\theta u_2 = (\epsilon\boldsymbol{\xi}^* + \boldsymbol{\Omega}^*)\cdot\nabla u_3, \tag{21b}$$

where ξ^* is the known vector $(0, \partial u_1/\partial r, -\partial u_1/\partial\theta)$ and $\mathrm{D}/\mathrm{D}t$ is the full advection operator

$$\frac{\partial}{\partial t} + u_1\frac{\partial}{\cos\theta\,\partial\phi} + u_2\frac{\partial}{\partial\theta} + u_3\frac{\partial}{\partial r}.$$

Therefore, in principle, given a zonal density gradient, the solution of the two coupled equations (21 *a*) and (21 *b*) can be obtained by integrating the 'forcing' (the buoyancy term) along known characteristics that are now the absolute vortex lines (the term $\epsilon\xi^* + \boldsymbol{\Omega}^*$) in the meridional–vertical plane. The effect of the zonal velocity field u_1 is to shape the absolute vorticity field that allows the remaining velocity component to be determined. We believe that this is not the best way to proceed, however, because the velocity field must satisfy the continuity equation as well. A more direct approach based on the momentum equations that solves this difficulty will be given in §4. Kinematically, knowledge of the vorticity field in an incompressible fluid determines the velocity field up to a potential flow. Here, the vorticity components ξ_2 and ξ_3 are determined from knowledge of u_1 while the smaller component ξ_1 is still unknown. The effect of the anisotropic approximation has been to single out the zonal velocity component u_1 as the only rotational component. As a consequence, at this order of approximation, the velocity components u_2 and u_3 derive from a potential and may be easily determined from the continuity equation, taking the term $(1/a\cos\theta)(\partial u_1/\partial\phi)$ in that equation as a known term. This is just the transport in the meridional–vertical plane that is necessary to accommodate the divergences and convergences of the zonal velocity field (see §4).

It is both the small aspect ratio and anisotropy of equatorial flows that allows the result (19*a*, *b*). This discussion should be reminiscent of the oceanic situation that prevails at mid-latitudes near a coast. For similar reasons of anisotropy, the velocity component along the coast can be determined with geostrophy from the density gradient normal to the coast.

Given the approximated vorticity equations, it is a logical step to try to infer back the equivalent momentum equations at the same level of accuracy. The previous approximations leading to (19*b*) and (21*a*, *b*) are local approximations valid at low latitudes in the limit of small α. Because these equations encompass the mid-latitude geostrophic prescriptions in situations of small ϵ and have therefore global value on the sphere, it would be tempting to obtain momentum equations whose curl gives back the approximated vorticity equations. Unfortunately the small-α approximation makes a selection among nonlinear terms and it is not possible to do that in spherical coordinates because unavoidably spurious metric type terms of $O(\gamma)$ are generated. For this reason, we present the local momentum equations valid at the equator to leading order in γ, i.e. on the equatorial β-plane. This involves the following expansions:

$$\left.\begin{aligned} \sin\theta = \gamma y + O(\gamma^3), \quad \cos\theta = 1 + O(\gamma^2), \\ x = a\phi, \quad y = a\theta, \quad z = (r-a), \end{aligned}\right\} \tag{22}$$

where the Cartesian coordinates x, y and z now replace the spherical ϕ, θ and r respectively.

Under such conditions, the appropriate momentum equations are

$$\epsilon\frac{\mathrm{D}u_1}{\mathrm{D}t} - \gamma y\, u_2 + \delta u_3 = -P_x, \tag{23a}$$

$$\gamma y\, u_1 = -P_y, \tag{23b}$$

$$-\delta u_1 = -P_z - \rho, \tag{23c}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + u_1\,\partial/\partial x + u_2\,\partial/\partial y + u_3\,\partial/\partial z$.

If the curl (in Cartesian coordinates) of (23) is taken, one recovers exactly the vorticity equations (19*b*), (21*a*, *b*) once the approximations (22) have been made in those equations. Full compatibility is therefore ensured at $O(\gamma)$ between the two sets, the relative vorticity vector in the new Cartesian coordinates being $(0, \partial u_1/\partial z, -\partial u_1/\partial y)$, a form that also preserves the solenoidality condition for vorticity.

To the same order of approximation, the angular momentum (13), satisfying its conservation principle, can be rewritten as

$$\Gamma = 1 - \underset{\text{(I)}}{\gamma^2\theta'^2} + \underset{\text{(II)}}{2\delta\gamma r'} + \underset{\text{(III)}}{2\epsilon\gamma u_1} \tag{24}$$

or in dimensional form as

$$\Omega a^2(1-\theta^2) + 2\Omega a(r-a) + au_1.$$

Equation (24) shows that the radially varying term (II) has to be kept compared to the latitudinal-dependent term (I) as long as $2\delta/\gamma$ is of order one. This result can be understood when looking at the geometry of the equatorial region. In such an equatorial band, because the horizontal plane aligns with the Earth's rotation axis, the latitudinal change of planetary angular momentum in moving a fluid particle over a distance L becomes weak and approaches the magnitude of its radial change caused by the displacement of the particle over a depth H. That proscribes, therefore, the vertical displacement over those scales to be freed from the planetary rotational constraint.

It is perhaps useful to rewrite the above set in dimensional form:

$$\frac{\mathrm{D}u_1}{\mathrm{D}t} - \beta y\, u_2 + 2\Omega u_3 = -\frac{P_x}{\rho_0}, \tag{23d}$$

$$\beta y\, u_1 = -\frac{P_y}{\rho_0}, \tag{23e}$$

$$-2\Omega u_1 = -\frac{P_z}{\rho_0} - g\frac{\rho}{\rho_0}. \tag{23f}$$

We emphasize that this set is appropriate to discuss low-frequency finite-amplitude equatorial flows in the limit of small meridional scales and long zonal scales, thereby excluding zonal boundary regions. All Coriolis terms involved with the horizontal component of the Earth's rotation must be kept, not only for consistency with the vorticity analysis but also for consistency with energy considerations. The appropriate energy equation is obtained by multiplying scalarly (23) by the velocity field, leading to

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\rho_0 u_1^2}{2}\right) = -\boldsymbol{u}\cdot\boldsymbol{\nabla}P - \rho g u_3.$$

Comparing (23) with the linear set (5) mentioned in the introduction, we see that the zonal acceleration term must be included. Equatorial dynamics are frequently studied by taking advantage of the hydrostatic approximation. However, the elimination of the velocity between (23e) and (23f) shows that

$$2\Omega\frac{\partial P}{\partial y} + \beta y\frac{\partial P}{\partial z} = -\beta y\, g\rho,$$

an expression which can be rewritten as

$$\mathrm{d}P/\mathrm{d}s = -g\rho\theta, \tag{23g}$$

where s is a coordinate along the true Earth's rotation vector. The correct relation between pressure and density (23g) relates pressure variations along the rotation axis to the projection of the buoyancy vector onto that axis. This reduces to the statement of hydrostatic pressure for precisely the same condition found previously, i.e. when flows are broader than the horizontal scale $(Ha)^{1/2}$.

4. A generalization of the geostrophic method

The vorticity equation is a differentiated by-product of the momentum equations. Determination of velocity from density therefore requires an integration assuming as initial data a known velocity somewhere (a method currently used in ocean studies to infer the velocity field from observed density data, i.e. the integration of the thermal wind equations). An alternative way makes use of the pressure field, obtained either from hydrostatic dynamics or from observations in meteorology, to deduce the velocity field in applying geostrophic relations. These two methods use either the simplified equation (3) or the simplified set (1) or (2) which are mutually consistent at mid-latitudes but not in the vicinity of the equator. While the preceding sections dealt with the validity of the approximations for mid- and low-latitudes, the following extends formally these results in the limit of low-frequency motions using the full momentum equations. We show how to recover the previous results from a very general vectorial relationship that relates velocity to density up to an arbitrary constant of integration. The formula is then tailored to a spherical geometry and exploited in mid- and low-latitude regions to discuss the hydrostatic pressure hypothesis currently made in the study of large-scale geophysical flows.

Because both the velocity and vorticity vectors have no divergence, the vorticity equation (10) can be rewritten as

$$\nabla \times \left(\frac{\partial \boldsymbol{u}}{\partial t} + \xi_a \times \boldsymbol{u} - \boldsymbol{g}\frac{\rho}{\rho_0} \right) = 0, \tag{25}$$

where $\xi_a = \xi + 2\boldsymbol{\Omega}$ is the absolute vorticity. A vector of zero curl being the gradient of a scalar potential χ, it is possible to write

$$\frac{\partial \boldsymbol{u}}{\partial t} + \xi_a \times \boldsymbol{u} - \boldsymbol{g}\frac{\rho}{\rho_0} = -\nabla \chi, \tag{26}$$

where χ is none other than the dynamic pressure $(p/\rho_0 + \frac{1}{2}|\boldsymbol{u}|^2)$.

Of course χ is unknown so that it is not obvious that the momentum formulation will be useful. If we find a way to determine χ and if some useful insight helps to determine the absolute vorticity field, the steady version of (26) shows that the computation of $\boldsymbol{u}$ amounts to a simple evaluation. Quite remarkably, the relation derived below does just this. Suppose indeed that the flow that is sought is either steady, or has a frequency ω much lower than 2Ω (assumed to be an appropriate measure of the absolute vorticity amplitude). Projecting (26) onto the absolute vorticity vector ξ_a gives

$$\frac{\rho}{\rho_0}\boldsymbol{g}\cdot\xi_a = \xi_a \cdot \nabla \chi. \tag{27}$$

In this expression valid to $O(\omega/2\Omega)$, the velocity field has been eliminated and the dynamic pressure χ may therefore be obtained from the density field if the absolute vorticity vector is known. When this is the case, (27) becomes

$$\frac{\mathrm{d}\chi}{\mathrm{d}s} = \frac{\rho}{\rho_0}\frac{\boldsymbol{g}\cdot\xi_a}{|\xi_a|}$$

or
$$\frac{d\chi}{ds} = \frac{\rho}{\rho_0} g \cos\alpha, \tag{28}$$

where s designates the curvilinear coordinate along absolute vortex lines and α is the angle between the gravity vector $\boldsymbol{g}$ and the absolute vorticity vector ξ_a. It is then a simple integration along characteristics (the absolute vortex lines) to obtain χ up to a constant χ_0 given at some initial position s_0. After χ is determined, the low-frequency version of (26) becomes simply

$$\xi_a \times \boldsymbol{u} = \boldsymbol{g}\frac{\rho}{\rho_0} + \nabla\left(\int_{s_0}^{s} \frac{\rho}{\rho_0} g \cos\alpha \, ds\right) + \nabla\chi_0. \tag{29}$$

Equations (28) and (29) can be viewed as a set of 'generalized' hydrostatic and geostrophic relations.

With the low-frequency approximation, $\boldsymbol{u}$ can be obtained everywhere from (29) if the initializing function χ_0 is known and if the forcing vector on the right-hand side is not parallel to ξ_a. Of course this is purely formal at this stage because the absolute vorticity vector is not known in general and depends itself on the unknown velocities. Furthermore the potential χ can be obtained from the density field only if the angle α between the gravity vector and the absolute vorticity vector is accurately known. Progress can be made if further assumptions are made about the shape of the absolute vortex lines.

The scaling analysis presented previously for slow predominantly zonal flows at low latitudes corresponds to the choice of an absolute vorticity vector in the meridional, vertical plane only:

$$\xi_a = \left(0, \frac{\partial u_1}{\partial r} + 2\Omega\cos\theta, -\frac{1}{r}\frac{\partial u_1}{\partial \theta} + 2\Omega\sin\theta\right).$$

It has been shown in §3 how to determine u_1 from meridional density gradients up to an integration constant but we are now in a better position to discuss more fully the dynamics in the meridional–vertical plane. The determination of the flow in that plane may be carried out as follows. First decompose the velocity vector $\boldsymbol{u}^v = (u_2, u_3)$ as

$$\boldsymbol{u}^v = \nabla' V + \boldsymbol{i} \times \nabla'\psi,$$

where
$$\nabla' = \left(\frac{\partial}{r\partial\theta}, \frac{\partial}{\partial r}\right).$$

The continuity equation provides a two-dimensional Poisson's equation for V which can be solved given boundary conditions on V:

$$\nabla'^2 V = -\frac{1}{r\cos\theta}\frac{\partial u_1}{\partial\phi}. \tag{30}$$

Furthermore since the absolute vortex lines are known, the distribution of dynamic pressure χ can be obtained by integrating equation (28) along a known characteristic, introducing a second integration constant. Finally when χ is known, the zonal momentum equation provides an equation for ψ:

$$\xi_a' \cdot \nabla'\psi = -\frac{1}{r\cos\theta}\frac{\partial\chi}{\partial\phi} - \xi_a' \times \nabla' V. \tag{31}$$

The stream function ψ that recirculates fluid in the meridional–vertical plane is simply obtained by integrating once more the 'forcing' on the right-hand side of (31) along the absolute vortex lines, a calculation that involves a third constant of integration. With the addition of the boundary conditions on V, the determination of u_1, u_2, u_3 and χ from the density field requires four initial conditions for those same variables along some initial curve, the only restriction to be placed on that curve being that it can never be parallel to the Earth's rotation axis (for u_1) or to the absolute vortex lines (for the other variables). The obvious choice is that this curve be the equator itself. Given observations of all dynamic variables at the equator and of density in the interior, equations (19*a*), (28), (30) and (31) allow at least in principle a reduction of the determination of the velocity and pressure fields to four quadratures. We may expect that application of this procedure might be carried out iteratively: once the velocities are determined, an improved absolute vorticity field can be computed and the whole procedure started over again until convergence.

This general formulation complements the analysis of §3 about the conditions of validity of the traditional hydrostatic approximation for the mid- and for the low-latitudes cases respectively. Before doing so, we need to rewrite the low-frequency limit of (26) in component form:

$$(\xi_2+2\Omega\cos\theta)\,u_3-(\xi_3+2\Omega\sin\theta)\,u_2 = -\frac{1}{r\cos\theta}\frac{\partial\chi}{\partial\phi}, \tag{32a}$$

$$(\xi_3+2\Omega\sin\theta)\,u_1-\xi_1 u_3 = -\frac{1}{r}\frac{\partial\chi}{\partial\theta}, \tag{32b}$$

$$\xi_1 u_2-(\xi_2+2\Omega\cos\theta)\,u_1 = -\frac{\partial\chi}{\partial r}-\frac{g\rho}{\rho_0}. \tag{32c}$$

(i) *The hydrostatic approximation in the linear limit*

In the linear limit of infinitesimal amplitudes the absolute vorticity reduces to that of the Earth's component and the variations of dynamic pressure to those of static pressure, so that (28) and (29) become

$$\frac{1}{\rho_0}\frac{\mathrm{d}P}{\mathrm{d}s} = -\frac{\rho}{\rho_0}g\sin\theta, \tag{33a}$$

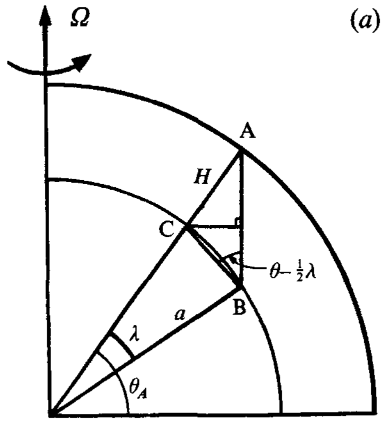
$$2\boldsymbol{\Omega}\times\boldsymbol{u} = \boldsymbol{g}\frac{\rho}{\rho_0}-\boldsymbol{\nabla}\left(\int_s^{s_0}\frac{\rho}{\rho_0}g\sin\theta\,\mathrm{d}s\right), \tag{33b}$$

where θ is the latitude and s_0 is the free surface where the pressure is assumed to be the constant atmospheric pressure. But $\mathrm{d}P/\mathrm{d}s$ is also $(\cos\theta/r)\,(\partial P/\partial\theta)+\sin\theta\,\partial P/\partial r$ so that (33a) becomes

$$\frac{1}{r}\frac{\partial P}{\partial\theta}+\tan\theta\left(\frac{\partial P}{\partial r}+\rho g\right) = 0. \tag{33c}$$

The above expression reduces to the familiar statement of hydrostatics when the first term on the left-hand side of (33*c*) is negligible. The necessary condition, $H/L\tan\theta$ much smaller than one, is a local condition and does not ensure hydrostatics combined with geostrophy to be valid over all the vertical scale H. To be so, one has to integrate the pressure along the Earth's rotation axis (see figure 4*a*) and show that the error induced by replacing

$$\int_{s(B)}^{s_0(A)}\frac{\rho}{\rho_0}g\sin\theta\,\mathrm{d}s$$



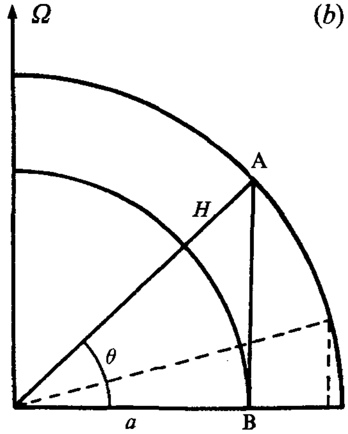


FIGURE 4. (*a*) Evaluation of the latitudinal extend $\lambda = \theta_A - \theta_B$ of a fluid column parallel to the rotation axis: the length BC is given by $H\cos(\theta_A)/\sin(\theta_A - \frac{1}{2}\lambda)$ and also by $2R\sin(\frac{1}{2}\lambda)$, leading to the geometrical relation (34). For hydrostatic balance to be valid, typical lengthscales L have to be greater than $a\lambda$. (*b*) Equatorial cases: $\theta_A = \theta = \lambda$, $\theta_B = 0$. Solid lines: hydrostatic relation valid over the full depth H of the ocean; dashed lines: hydrostatic relation valid only in the upper part of the ocean (see text).

with
$$\int_{r(C)}^{r_0(A)} \frac{\rho}{\rho_0} g \, \mathrm{d}r$$

in (33*b*) be small. This error in (33*b*) is small as long as the horizontal lengthscale L is much larger then the length $a\lambda$ of the arc subtended by the angle $\lambda = \theta_A(s_0) - \theta_B(s)$ which is given geometrically by

$$\sin(\tfrac{1}{2}\lambda)\sin(\theta_A - \tfrac{1}{2}\lambda) = \frac{H\cos(\theta_A)}{2a}, \tag{34}$$

a being the Earth's radius (see figure 4*a*).

At mid-latitudes, since H/a is very small, λ is small and very close to being $H/a\tan\theta$. The condition L larger than $a\lambda$ reduces then to the local condition ($H/L\tan\theta \ll 1$) with the angle θ effectively constant in (33*a*) and thus in (33*c*) over the full domain along the rotation axis. The local condition is sufficient to ensure hydrostatic balance for shallow fluids like the ocean or the atmosphere.

This demonstration shows that the hydrostatic relationship is valid to $O(H/L \tan\theta)$ and therefore can be used over a large range of scales. The result defines what is meant by mid-latitudes under geostrophic dynamics.

From the angular-momentum viewpoint, hydrostatics is valid if variations of Γ with radial distance (i.e. the vertical-velocity-related Coriolis term) are small compared to variations with latitude (i.e. the traditional Coriolis terms). In comparing term (II) to variations of term (I) in (13), the condition $\delta/\tan(\theta) \ll 1$ is recovered. Angular momentum can therefore be kept constant radially over the full depth of the fluid when moving particles over a distance L. By displacing the fluid over a depth H, the rotational constraint is not felt and only the gravity force and the vertical pressure gradient are experienced. This is precisely the condition that was found in § 3 to neglect the Coriolis terms associated with the horizontal component of the Earth's rotation.

However, within a narrow band of latitudes centred around the equator, the statement of hydrostatic pressures leads to a much more restrictive condition. When θ becomes small, the local approximation $L \gg H/a\theta$ gets increasingly worse because the segment AB spans a much larger range of latitudes and because the correct integration of the density field along AB in (33*b*) may then produce a pressure field sizeably different from the one obtained after a vertical integration. In this case (see figure 4*b*), θ_B being zero, the angle λ is of the same order as θ_A and is obtained from (34) by replacing θ_A by λ so that $\lambda \approx (2H/a)^{1/2}$. The condition L larger than $a\lambda$, to ensure that hydrostatic–geostrophic dynamics is valid over the depth H, becomes $L \gg (2Ha)^{1/2}$. This same condition is also obtained in considering term (II), in the angular momentum (24), to be smaller than term (I). Only in this case, do the classical hydrostatic–geostrophic relationships apply at the equator. They are valid over all the depth of the ocean if typical lengthscales of the motion are much greater than the arclength $a\lambda$ defined by the equatorial band shown in figure 4(*b*). For scales smaller than $(2Ha)^{1/2}$, new effects come from the neglected vertical Coriolis acceleration that plays a central role at the equator in imparting rigidity to fluid columns parallel to the Earth's rotation and not to the gravity axis. In other words, when the advection terms can be neglected, the thermal wind equations need to be applied to velocity components in the plane perpendicular to the rotation axis and not that perpendicular to gravity. At the angular-momentum level, the meridional distance, over which latitudinal variations of Γ become large, increases when approaching the equator, its radial variations can no longer be neglected if smaller lengthscales are considered. These results are in accordance with the vorticity viewpoint developed in the preceding chapters. Nevertheless, note that the traditional set is still valid over a finite depth in the upper part of the equatorial region. When θ or λ get small (figure 4*b*), hydrostatic balance can still be used over smaller horizontal scales when applied in the upper part of the ocean over a depth range H provided

$$H \leqslant L^2/2a.$$

(ii) *The hydrostatic approximation in the finite-amplitude regime*

Consider now the case of finite-amplitude flows at small but non-zero Rossby number. Again χ can be approximated by P at leading order. It is not difficult to show that (28) can be transformed to an expression similar to (33*a*) in which the latitude θ has to be replaced by the angle between the absolute vorticity vector and the horizontal plane. Henceforth provided that angle does not become small, the condition of small aspect ratio ensures local hydrostatics. At mid-latitudes the small-Rossby-number condition guarantees that the absolute vortex lines make a finite angle with the horizontal plane because the vertical vorticity is dominated by that of the Earth. The situation is nearly the opposite at low latitudes where the Earth's vorticity becomes tangent to the Earth's surface. Therefore, for the hydrostatic equilibrium to exist in the limit of small aspect ratio, the flow must be strong enough to reorient the absolute vortex lines away from the horizontal plane. As shown in the next section, the relative vorticity can easily turn the absolute vorticity vector in almost any direction in these regions. However, in the particular case of finite amplitude but anisotropic flow we have seen in §3 that the relation between pressure and density reduces to (33*a*) so that the above linear analysis is valid. In the more general case, expanding (28) gives

$$\frac{s_1}{r\cos\theta}\frac{\partial\chi}{\partial\phi}+\frac{s_2}{r}\frac{\partial\chi}{\partial\theta}+s_3\left(\frac{\partial\chi}{\partial r}+g\frac{\rho}{\rho_0}\right)=0,$$

where s_1, s_2 and s_3 are the components of the unit vector parallel to the absolute vorticity. It is clear that the last term vanishes if

$$\delta \leqslant (s_3/s_2, s_3/s_1\,\alpha).$$

If we can construct flow solutions where the absolute vorticity vector does not make a small angle with the horizontal plane then the following generalized statement of hydrostatics will be valid at order δ:

$$\frac{\partial\chi}{\partial r}=-\frac{\rho}{\rho_0}g.$$

Because it appears difficult to construct equatorial solutions where this condition is not violated at least somewhere in the flow and because it is highly desirable to use dynamical equations that span the whole range of flow amplitudes, we advise that the full Coriolis term be retained in the vicinity of the equator unless one has strong reason to believe that the flow evolves on lateral scales broader than $(Ha)^{1/2}$.

5. Equatorial jet

To conclude: we consider in this paragraph some special simple solutions in the equatorial region that may serve to illustrate some of the points made earlier. Let us look at the idealized situation of zonally homogeneous flows, when the density and pressure fields are independent of longitude. This case may perhaps be encountered both in the atmosphere and ocean away from boundary effects at low latitudes. Flows in the meridional–vertical plane must satisfy the zonal momentum equation (32*a*) that reduces to

$$\xi_{2a}u_3-\xi_{3a}u_2=0. \tag{35}$$

The trivial way to satisfy (35) is to have no flow in the meridional–vertical plane. In this purely zonal jet it is easy to show that the two other momentum equations, (32*b*) and (32*c*), can be combined to give

$$2\boldsymbol{\Omega}\cdot\nabla\frac{P}{\rho_0}=\frac{\rho}{\rho_0}\boldsymbol{g}\cdot 2\boldsymbol{\Omega}. \tag{36}$$

Although the relative vorticity of the jet can be of arbitrary amplitude, the relation between the pressure and density is linear, in complete agreement with the result (23*g*)

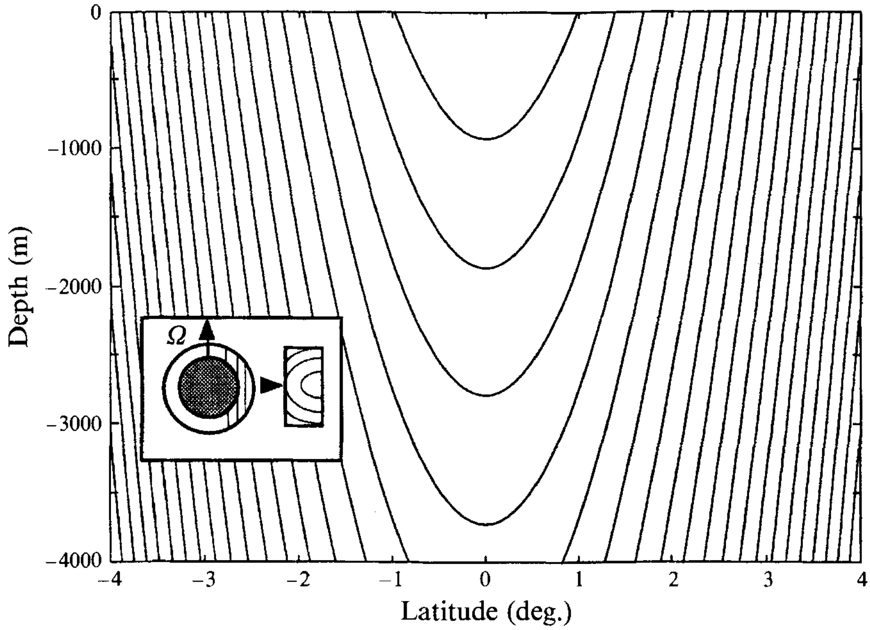


FIGURE 5. Lines of constant angular momentum $\Omega[r\cos(\theta)]^2$ in the equatorial region in the case when the zonal velocity vanishes. These lines are just parallel to the rotation axis. The inset sketches the way these lines are deformed when Cartesian coordinates are used.

of §3. If the zonal velocity is given, then the vorticity equation (19) allows one to compute the density field and (36) gives the pressure field. As a consequence precisely the same horizontal scale $(Ha)^{1/2}$ marks the transition between a hydrostatic jet and a non-hydrostatic jet.

However, flows in the meridional–vertical plane are not forbidden provided that absolute vortex lines and streamlines are parallel. Because the velocities and vorticities are non-divergent in that plane, stream functions exist for both fields and must coincide according to (35). The flows governed by (35) are free steady and of arbitrary amplitude. Of course they must satisfy the boundary condition that the flow be tangent to surfaces containing the fluid. When these are spherical, u_3 must vanish and we obtain the boundary condition

$$\xi_{3a} u_2 = 0.$$

Free meridional flow along the boundaries ($u_2 \neq 0$) is allowed provided that the absolute vortex lines are sufficiently deformed by the zonal flow to become tangent to the surface of the sphere, i.e. $\xi_{3a} = 0$. Just at the equator, no relative vertical vorticity is needed to accomplish this since the Earth's rotation vector is already tangent to the sphere. On the other hand, when the absolute vortex lines intersect the bounding surfaces no meridional flow is possible along the boundary.

Equation (35) has an integral form which can be simply obtained from the angular momentum principle (8). Since $\Gamma = \Omega r^2 \cos^2(\theta) + u_1 r \cos(\theta)$ is conserved along streamlines in the meridional plane when no zonal pressure gradient is present, vortex lines and streamlines in the meridional–vertical plane are also constant-Γ lines and can, therefore, be computed solely from knowledge of the zonal velocity. When $u_1 = 0$ (figure 5), the constant-Γ lines are parallel to the rotation axis, but become curved in Cartesian coordinates. Figure 6 shows the situation of an equatorial eastward zonal jet,

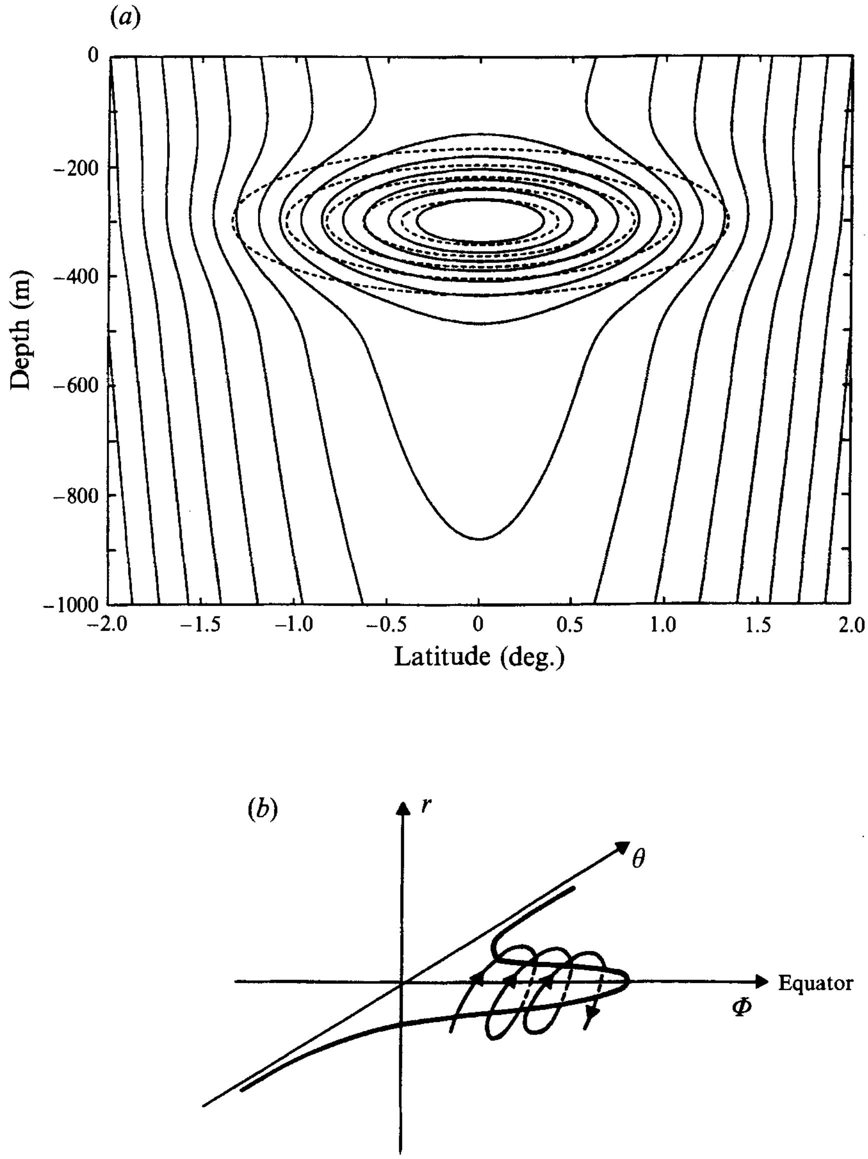


FIGURE 6. (*a*) The constant-angular-momentum lines (solid contours) in the meridional–vertical plane, between 0 and 1000 m depth, are pictured for the case of an oceanic eastward zonal jet centred at the equator. The jet normal to the plane (dashed lines: constant-zonal-velocity lines) has maximum speeds of 0.5 m s^{-1} around 300 m and e-folds over 1° in latitude and 100 m in the vertical. With such shears, well into the geophysical range, the relative vorticity can exceed and turn around the Earth's vorticity. When the absolute vortex lines are closed in the vicinity of the jet a steady inertial flow parallel to these solid contours is a solution of the equation of motion. (*b*) The trajectory of the fluid parcels are helices with generators parallel to the equator, winding themselves around the jet.

mimicking an oceanic situation, along with the associated constant-Γ lines or absolute vortex lines. The figure demonstrates that typical shears of observed equatorial jets can easily create absolute vorticity opposite to the Earth's vorticity $2\boldsymbol{\Omega}$. Near the equator this requires vertical shears of horizontal flow U/H to approach 2Ω or about 10 cm s^{-1} (1 m s^{-1}) over 700 m for the ocean (7000 m for the atmosphere). From the angular-momentum viewpoint, this means that contributions from relative flows (term (III) in

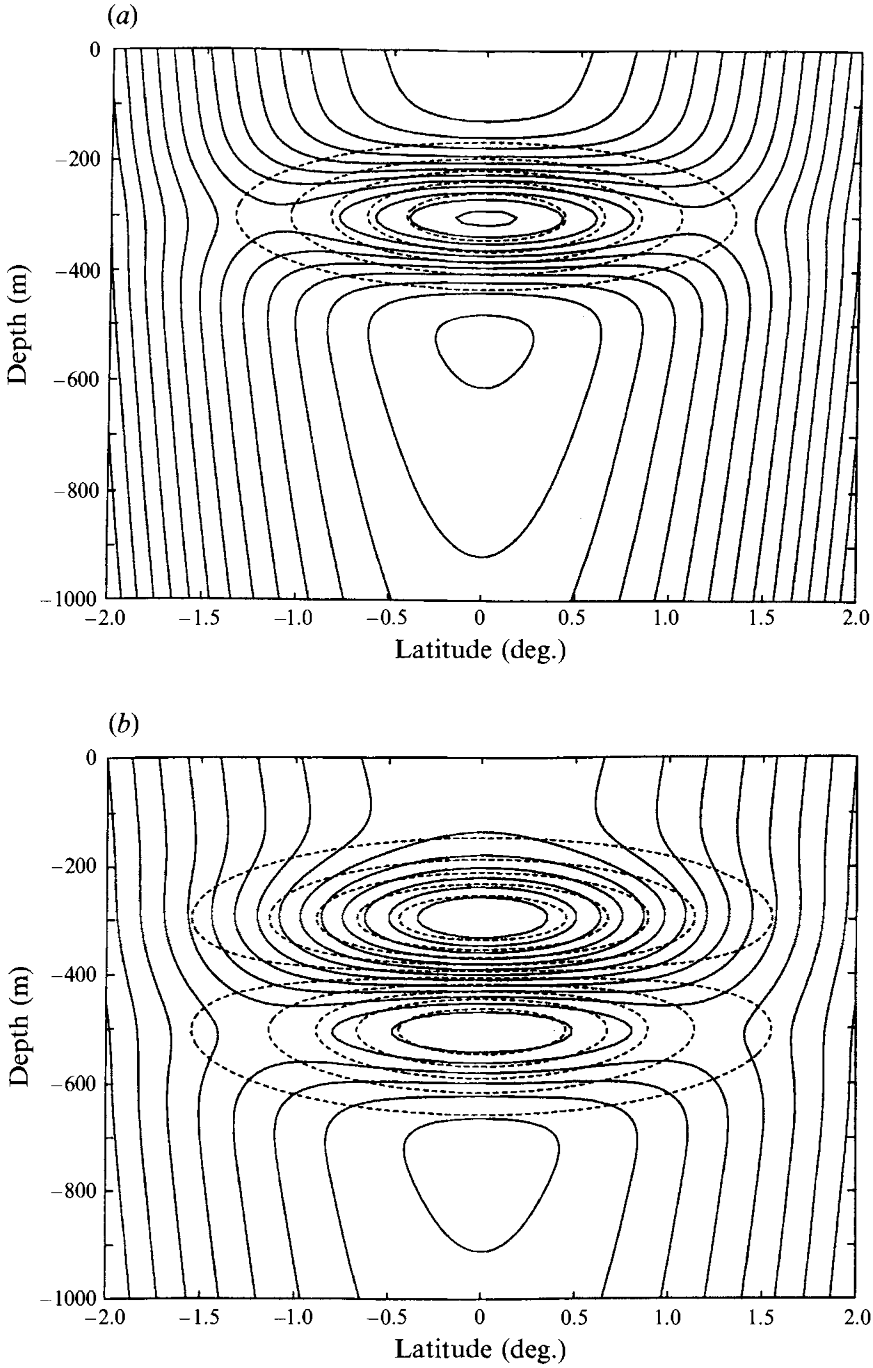


FIGURE 7. (*a*) Same as figure 6(*a*), but for a westward jet. One can observe a tightening of the lines outside the jet and a closing of contours underneath the jet region. (*b*) Superposed opposite jets with equal strength (eastward jet over westward jet): fewer contours are closed inside the westward jet since a negative zonal velocity weakens the planetary angular momentum.

(24)) are of the order of the radially dependent component (term II in (24)). Because such values are in the geophysical observed range, one may expect this situation of 'north to south' absolute vorticity to be rather common. In the vicinity of a jet, the absolute vortex lines close on themselves and *free steady inertial flows of arbitrary amplitude in the meridional–vertical plane are possible* if they are everywhere tangent to the absolute vortex lines. As one moves away from the jet, the simple Earth's rotation direction is recovered since u_1 tends to zero. In such flows, the three-dimensional trajectories of fluid particles are helices with generators parallel to the Equator. Figure 7 mimics now a westward jet. In contrast to the eastward case fewer contours are closed, negative velocities being able to weaken the magnitude of angular momentum. The reverse is true underneath and above the jet region. As one moves away from the equator, the latitudinal dependent term in the angular momentum becomes the dominant term and constant-Γ lines become parallel to the rotation axis as illustrated in figure 8. It is therefore much more difficult to observe the above flow solutions when the jet axis lies a small distance off the equator.

Because the whole flow solution is independent of longitude, the exact relation (28) shows that after a circuit in the meridional–vertical plane, the dynamic pressure must return to the same value. An integration of (28) in a situation of closed contours requires

$$\oint g \frac{\rho}{\rho_0} \cos(\alpha)\, \mathrm{d}s = 0.$$

This integral is simply the circulation of the vector $\rho \boldsymbol{g}$ along a closed curve. In such a quasi-steady situation, the integrated work of the gravity force must vanish to prevent changes in the circulation of the velocity field itself. Under adiabatic conditions, the density is also constant along a fluid trajectory and when this is the case, the above integral reduces to the circulation of $\boldsymbol{g}$ and is therefore zero (since $\boldsymbol{g}$ derives from a potential).

An investigation of the stability character of such angular momentum and density stratification has not been undertaken in this paper. Nevertheless one possible outcome in a situation of small turbulent mixing could be the following. When the angular momentum contours are closed, such inertial circulations are compatible with static stability of density, in the hydrostatic sense, only if the density is homogeneous inside the closed regions. Nevertheless, should the Coriolis force or inertial terms in the vertical be taken into account, vertical positive shearing motions are capable of reversing density gradients without losing their stable character, heavier fluid being expelled through those 'centrifugal' accelerations to shallower levels overlaying lighter fluid.

The zonal-homogeneity assumption implies bidimensionality of the flow in the meridional–vertical plane since the velocity u_1 normal to that plane depends only on coordinates in that plane. As a result, if ψ designates the stream function, the steady tracer equation for density, assuming Fickian turbulent mixing of intensity K, is

$$J(\psi, \rho) = K\nabla'^2 \rho,$$

where J is the appropriate two-dimensional Jacobian. When the flow is quasi-conservative so that the Péclet number UL/K is large, the situation for the tracer is similar to that for vorticity with closed streamlines in two dimensions. Batchelor's (1956) theorem states that at high Reynolds number, the vorticity must homogenize inside closed streamlines. This prediction implies, here, homogenization of density (or other tracers) inside the closed orbits. Such free quasi-conservative flows have zero

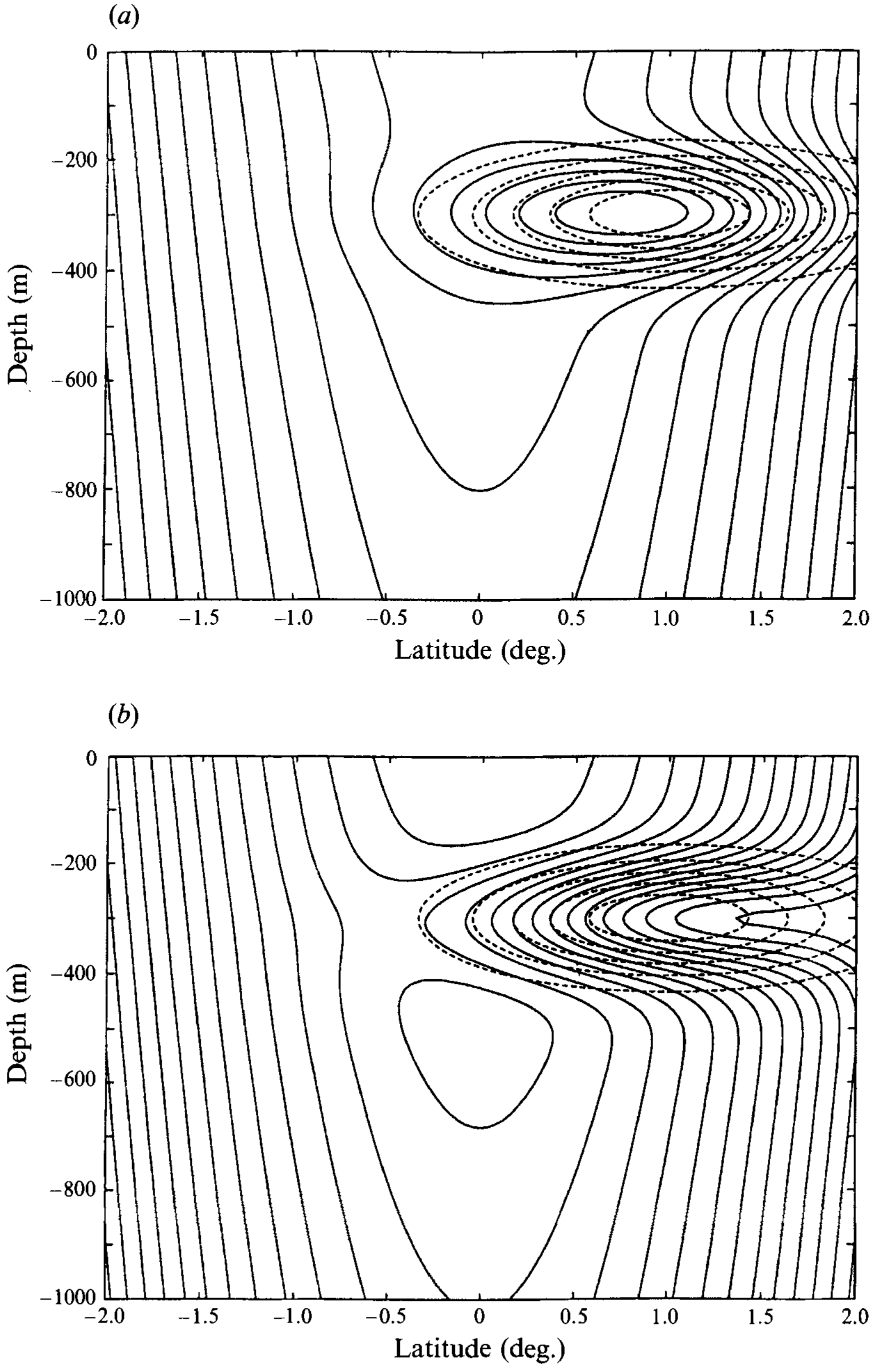


FIGURE 8. Same as figure 6(*a*), but with a latitudinal 1° shift (*a*) of the eastward-moving jet and (*b*) of the westward jet. Closed contours disappear rapidly while moving a westward jet to higher latitudes.

Ertel potential vorticity since the vectors ξ_a and $\nabla\rho$ are perpendicular in this situation. Under these special zonally homogeneous dynamics, a look at the density distribution in the fluid could well give indications of the absolute vortex lines or constant angular momentum lines and therefore of possible free paths for fluid exchanges across the equator without breaking any rotational constraints.

6. Summary

Within the context of the atmosphere, previous studies have discussed 'balanced dynamics' in the tropics but all of them rest on the *a priori* choice of hydrostatic balance in the vertical whose main purpose in meteorological models is to filter out acoustic waves. Instead, it is shown here that a better description of the zonal flow can be obtained if the hydrostatic approximation is abandoned. The relation (19), which is now free from singularities at the equator, relates the buoyancy torque in the zonal direction to the twisting effects of the full Earth's rotation vector by the zonal flow. The transition from such equatorial dynamics to classical mid-latitude geostrophy depends upon the value of the parameter $\tan\theta/\delta$ (δ being the ratio of vertical over horizontal scale). At mid-latitudes it is large and the traditional approximation of neglecting the Coriolis force associated with the horizontal component of the Earth's rotation can be made so that the pressure is hydrostatic. Near the equator this approximation requires the flow to evolve on scales larger than $(Ha)^{1/2}$. When shear is found in oceans or atmospheres on such scales, the present analysis definitely suggests that the traditional approximation and the hydrostatic assumption should be abandoned. Instead the mid-latitude thermal wind is replaced in such regions by a vorticity equilibrium in the zonal direction between the buoyancy torque and the twisting effect on the full Earth's vorticity, a relation that is valid at finite amplitude provided the speed of the zonal flow is much larger than the meridional one. When the density field is integrated along the Earth's rotation axis, the zonal flow is obtained up to an arbitrary constant in a manner reminiscent of current practice at mid-latitudes. What is particularly important is that the zonal flow sets the absolute vorticity field to leading order. At the momentum level, this means that all Coriolis terms in the meridional and vertical equations must be kept although some of the nonlinear advection terms can be left out. The low-latitude and mid-latitude approximations have been shown to stem from a rather general result (28) and (29) of the steady Euler equation that allows the velocity field to be formally computed provided the density and absolute vorticity field are known. Using this result, given an initialization of all flow variables at the equator, it is possible at least in principle to compute the values of these variables in the interior solely from observations of the density field and initial guesses for the vorticity.

At sufficient distances from solid boundaries, if the zonal flow becomes independent of longitude, the possible existence of steady inertial closed loops in the meridional–vertical plane has been investigated. This recirculation of fluid around equatorial jets is possible if the jet is strong enough to create closed contours of absolute vorticity. Streamlines in this meridional recirculation plane have been simply inferred from the use of the angular-momentum conservation principle (8). The number $U/2\Omega H$ ($=\epsilon/\delta$) must then reach at least order one, a situation that is rather common in geophysical flows.

In an oceanic context the relations (23*b*) and (23*c*) have been used experimentally by Joyce, Lukas & Firing (1988) to derive the transport of the equatorial undercurrent in the Central Pacific from hydrographic data and shown by these authors to improve the comparison between inferred and observed velocities. Given the broad conditions under which it is valid in the tropics, this relation should be further tested experimentally in both oceans and atmospheres. The latter case is specially appropriate because independent observations of zonal velocity and temperature are routine, making a test of the equatorial thermal wind equation possible. Below the tropopause the equatorward temperature gradient (along pressure surfaces) induces a negative zonal buoyancy torque in the Northern hemisphere. This is normally equilibrated by the positive vorticity tendency caused at mid-latitudes by the twisting of the vertical component of the Earth's rotation vector by the westerlies increasing aloft. Closer to the equator, equation (19) suggests an alternative due to the *horizontal* shear of the zonal wind acting on the *horizontal* component of the Earth's rotation: trade winds speeding up at lower latitudes can create just the right effect.

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